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Canonical transformations and general soliton solutions of some multidimensional field equations

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Abstract. For a kind of n-dimensional field equation $\partial_\alpha\partial_\alpha\varphi = \mathrm{d}F(\varphi)/\mathrm{d}\varphi$, let (q_i, p_i) and (φ_i, π_i) for $i = 1, 2$ be two sets of field variables. Consider a canonical transformation in the form $p_i = \int \mathrm{d}^n x\, \mathscr{P}_i(\varphi, \pi)$ and $\varphi_i = \varphi_i(q, \mathscr{P})$. It is shown that $\mathscr{P}_i(\varphi, \pi)$ are constants of motion, $\varphi_i(q, \mathscr{P})$ are the general soliton solutions for select $q_i(x)$, and the direct condition of canonical transformation leads to the Bäcklund transformation between two general soliton solutions. Some examples of multidimensional solitons are discussed in detail.

1. Introduction

The multidimensional nonlinear scalar field equations come from some important physical phenomena. For example, the studies of superconductivity brought us the Josephson equation and the behaviour of particles led to the Klein–Gordon equation. Therefore solving these equations is interesting and useful work. In order to do this work, the researchers have advanced many methods such as inverse scattering [1–3], the Bäcklund transformation [4, 5], the Riemann problem [6, 7], the Hirota method [8], the Gibbon method [9, 10] and the projection matrix [11, 12]. In a previous paper [13], we presented the method of canonical transformation for solving these equations.

Because the nonlinearity and the multidimension caused many difficulties for the researchers, so far only some specific solutions of the equations have been obtained [1–16]. In general, these specific solutions cannot satisfy the conditions of actual boundary values and initial values. In the present paper, we obtain a kind of general soliton solution of the equations, by applying the canonical transformations. These soliton solutions will be quite useful for concrete physical problems.

Throughout the paper we adopt a summation convention for repeated indices: a Greek index run from 0 to $n-1$, the other index runs from 1 to $n-1$ unless it is particularly stated otherwise.

2. Canonical transformations of field variables

Under the summation convention the multidimensional nonlinear scalar field equations are generally expressed as

$$\partial_\alpha\partial_\alpha\varphi_i = \mathrm{d}F(\varphi_i)/\mathrm{d}\varphi_i \qquad \partial_\alpha = \partial/\partial x_\alpha$$
$$x_0 = it \qquad x_1 = x \qquad x_2 = y \qquad x_3 = z, \ldots \tag{1}$$

For brevity we consider only two components of field variables, that is $i=1$ and 2. Let the conjugate canonical momentum of field variables φ_i be $\pi_i=\partial\varphi_i/\partial t$. The equation (1) has an equivalent form [17]

$$\frac{\partial\varphi_i}{\partial t}=\frac{\delta H}{\delta\pi_i} \qquad \frac{\partial\pi_i}{\partial t}=-\frac{\delta H}{\delta\varphi_i} \tag{2}$$

where H denotes the Hamiltonian. In our previous work [13], we obtained another equivalent form of (1), namely the direct condition of canonical transformation

$$\frac{\partial\varphi_i}{\partial q_j}=\frac{\delta p_j}{\delta\pi_i} \qquad \frac{\partial\pi_i}{\partial q_j}=-\frac{\delta p_j}{\delta\varphi_i} \tag{3}$$

where p_j and q_j are a pair of new conjugate canonical variables of fields. The transformation from (φ, π) to (q, p) is a canonical transformation of fields.

We take the canonical transformation in such a form

$$\begin{aligned} &\varphi_i=\varphi_i(q_1, q_2) \qquad \pi_i=\pi_i(q_1, q_2) \\ &p_i[\varphi(x), \pi(x)]=\int \mathrm{d}^{n-1}x\, \mathscr{P}_i(\varphi, \pi) \end{aligned} \tag{4}$$

that (1) and (3) become respectively

$$\frac{\partial^2\varphi_i}{\partial q_1\partial q_2}=\frac{\mathrm{d}F}{\mathrm{d}\varphi_i}. \tag{5}$$

Inserting (4) into (1) yields

$$\partial_\alpha\partial_\alpha\varphi_i=\partial_\alpha\partial_\alpha q_1\frac{\partial\varphi_i}{\partial q_1}+\partial_\alpha\partial_\alpha q_2\frac{\partial\varphi_i}{\partial q_2}+\partial_\alpha q_1\partial_\alpha q_1\frac{\partial^2\varphi_i}{\partial q_1^2}+\partial_\alpha q_2\partial_\alpha q_2\frac{\partial^2\varphi_i}{\partial q_2^2}+2\partial_\alpha q_1\partial_\alpha q_2\frac{\partial^2\varphi_i}{\partial q_1\partial q_2}=\frac{\mathrm{d}F}{\mathrm{d}\varphi_i}. \tag{6}$$

Combining (5) with (6) we have the system of equations

$$\begin{aligned} &\partial_\alpha\partial_\alpha q_1=0 \qquad \partial_\alpha q_1\partial_\alpha q_1=0 \\ &\partial_\alpha\partial_\alpha q_2=0 \qquad \partial_\alpha q_2\partial_\alpha q_2=0 \end{aligned} \qquad \partial_\alpha q_1\partial_\alpha q_2=\tfrac{1}{2}. \tag{7}$$

By substituting (4) into (3), one obtains

$$\frac{\partial\varphi_i}{\partial q_j}=\frac{\partial\mathscr{P}_j}{\partial\pi_i} \qquad \frac{\partial\pi_i}{\partial q_j}=-\frac{\partial\mathscr{P}_j}{\partial\varphi_i}. \tag{8}$$

Given (8) and (5), we may obtain the equation of the densities of momenta as

$$\frac{\partial\mathscr{P}_k}{\partial\pi_j}\frac{\partial^2\mathscr{P}_l}{\partial\varphi_j\partial\pi_i}-\frac{\partial\mathscr{P}_k}{\partial\varphi_j}\frac{\partial^2\mathscr{P}_l}{\partial\pi_j\partial\pi_i}=\frac{\mathrm{d}F}{\mathrm{d}\varphi_i} \qquad j=1,2 \qquad k\neq l. \tag{9}$$

The equations (7) and (9) determine the new canonical variables (q and p).

Equation (7) is two d'Alembert equations with the conditions $\partial_\alpha q_1\partial_\alpha q_1=\partial_\beta q_2\partial_\beta q_2=0$, $\partial_\alpha q_1\partial_\alpha q_2=\frac{1}{2}$. Consider some general solutions of (7) as

$$\begin{aligned} &q_i=f_i(\eta_k)+b_{i\alpha}x_\alpha \qquad \eta_{k\alpha}=a_{k\alpha}x_\alpha+\delta_k \\ &\delta_k=\text{constant} \qquad k=1,2,\ldots,N \end{aligned} \tag{10}$$

where $f_i(\eta_k)$ are some arbitrary functions of η_k. Applying (10) to (7) leads to

$$\partial_\alpha q_i \partial_\alpha q_j = a_{k\alpha} a_{l\alpha} \frac{\partial f_i}{\partial \eta_k} \frac{\partial f_j}{\partial \eta_l} + a_{k\alpha} b_{j\alpha} \frac{\partial f_i}{\partial \eta_k} + a_{k\alpha} b_{i\alpha} \frac{\partial f_j}{\partial \eta_k} + b_{i\alpha} b_{j\alpha} = \begin{cases} 0 & (i=j) \\ \frac{1}{2} & (i \neq j) \end{cases}$$

$$\partial_\alpha \partial_\alpha q_i = a_{k\alpha} a_{l\alpha} \frac{\partial^2 f_i}{\partial \eta_k \partial \eta_l} = 0 \qquad k, l = 1, 2, \ldots, N \qquad i, j = 1, 2. \tag{11}$$

The arbitrariness of functions $f_i(\eta_k)$ makes the constants $a_{k\alpha}$, $b_{i\alpha}$ obey the equations

$$a_{k\alpha} a_{l\alpha} = 0 \qquad a_{k\alpha} b_{j\alpha} = 0 \qquad b_{i\alpha} b_{j\alpha} = \begin{cases} 0 & (i=j) \\ \frac{1}{2} & (i \neq j). \end{cases} \tag{12}$$

Equation (12) implies $\frac{1}{2}(N+2)(N+3)$ equations with $(N+2)n$ constants $a_{k\alpha}$, $b_{i\alpha}$ for $\alpha = 0, 1, \ldots, n-1$; $k = 1, \ldots, N$, $i = 1, 2$. Therefore the number N of variables η_k must satisfy the inequality $\frac{1}{2}(N+2)(N+3) \leqslant (N+2)n$, that is

$$N \leqslant 2n - 3. \tag{13}$$

Equation (9) contains two nonlinear equations. In general, to solve them is difficult work. However, we can easily obtain some of their simple specific solutions. For instance setting $\mathscr{P}_i$ with separate variables as [18]

$$\mathscr{P}_i = f_i(\varphi_1) + F_i(\varphi_2) + g_i(\pi_1) + G_i(\pi_2) \tag{14a}$$

or

$$\mathscr{P}_i = f_i(\varphi_1) g(\pi_1) + F_i(\varphi_2) G(\pi_2) \tag{15a}$$

we insert (14*a*) and (15*a*) into (9), respectively, obtaining the specific solutions

$$\begin{aligned} \mathscr{P}_1 &= \tfrac{1}{2} C_1 \pi_1^2 + \tfrac{1}{2} C_2 \pi_2^2 - C_3^{-1} F(\varphi_1) - C_4^{-1} F(\varphi_2) \\ \mathscr{P}_2 &= \tfrac{1}{2} C_3 \pi_1^2 + \tfrac{1}{2} C_4 \pi_2^2 - C_1^{-1} F(\varphi_1) - C_2^{-1} F(\varphi_2) \end{aligned} \tag{14b}$$

$$\begin{aligned} \mathscr{P}_1 &= \sin \pi_1 \sqrt{2 C_1 F(\varphi_1)} + \cos \pi_2 \sqrt{2 C_2 F(\varphi_2)} \\ \mathscr{P}_2 &= \sin \pi_1 \sqrt{2 C_1^{-1} F(\varphi_1)} + \cos \pi_2 \sqrt{2 C_2^{-1} F(\varphi_2)} \end{aligned} \tag{15b}$$

and so on, where C_i for $i = 1, 2, 3, 4$ are constants.

Furthermore we take (14*b*) as an example to find the solutions $\varphi_i = \varphi_i(q_1, q_2)$ of the canonical transformation (4). From (14*b*) we have

$$\begin{aligned} \pi_1 &= \left\{ \frac{2}{C_2 C_3 - C_1 C_4} \left[C_2 \mathscr{P}_2 - C_4 \mathscr{P}_1 - \left(\frac{C_4}{C_3} - \frac{C_2}{C_1} \right) F(\varphi_1) \right] \right\}^{1/2} \\ \pi_2 &= \left\{ \frac{2}{C_2 C_3 - C_1 C_4} \left[C_3 \mathscr{P}_1 - C_1 \mathscr{P}_2 - \left(\frac{C_1}{C_2} - \frac{C_3}{C_4} \right) F(\varphi_2) \right] \right\}^{1/2}. \end{aligned} \tag{16}$$

We may prove $\mathscr{P}_1$ and $\mathscr{P}_2$ are two constants of motion, since from (4) and (8) we have

$$\begin{aligned} \dot{\mathscr{P}}_i &= \frac{\partial \mathscr{P}_i}{\partial \varphi_j} \frac{\partial \varphi_j}{\partial q_k} \frac{\partial q_k}{\partial t} + \frac{\partial \mathscr{P}_i}{\partial \pi_j} \frac{\partial \pi_j}{\partial q_k} \frac{\partial q_k}{\partial t} \\ &= \left(\frac{\partial \mathscr{P}_i}{\partial \varphi_j} \frac{\partial \mathscr{P}_k}{\partial \pi_j} - \frac{\partial \mathscr{P}_i}{\partial \pi_j} \frac{\partial \mathscr{P}_k}{\partial \varphi_j} \right) \frac{\partial q_k}{\partial t} = 0 \qquad i, j, k = 1, 2. \end{aligned} \tag{17}$$

Setting the constants as

$$\left[\frac{2}{C_2C_3-C_1C_4}(C_2\mathscr{P}_2-C_4\mathscr{P}_1)\right]^{1/2}=a_1 \qquad \left[\frac{2}{C_2C_3-C_1C_4}\left(\frac{C_4}{C_3}-\frac{C_2}{C_1}\right)\right]^{1/2}=b_1$$
$$\left[\frac{2}{C_2C_3-C_1C_4}(C_3\mathscr{P}_1-C_1\mathscr{P}_2)\right]^{1/2}=a_2 \qquad \left[\frac{2}{C_2C_3-C_1C_4}\left(\frac{C_1}{C_2}-\frac{C_3}{C_4}\right)\right]^{1/2}=b_2 \tag{18}$$

(16) is simplified to

$$\pi_1=\sqrt{a_1-b_1F(\varphi_1)} \qquad \pi_2=\sqrt{a_2-b_2F(\varphi_2)}. \tag{19}$$

The application of (8), (14*b*) and (19) gives

$$\mathrm{d}\varphi_1=\frac{\partial\varphi_1}{\partial q_i}\mathrm{d}q_i=\frac{\partial\mathscr{P}_i}{\partial\pi_1}\mathrm{d}q_i$$
$$=C_1\pi_1\,\mathrm{d}q_1+C_3\pi_1\,\mathrm{d}q_2=\sqrt{a_1-b_1F(\varphi_1)}(C_1\,\mathrm{d}q_1+C_3\,\mathrm{d}q_2)$$
$$\mathrm{d}\varphi_2=\frac{\partial\varphi_2}{\partial q_i}\mathrm{d}q_i=\frac{\partial\mathscr{P}_i}{\partial\pi_2}\mathrm{d}q_i$$
$$=C_2\pi_2\,\mathrm{d}q_1+C_4\pi_2\,\mathrm{d}q_2=\sqrt{a_2-b_2F(\varphi_2)}(C_2\,\mathrm{d}q_1+C_4\,\mathrm{d}q_2).$$

Setting $C_1=d_{11}$, $C_3=d_{12}$, $C_2=d_{21}$, $C_4=d_{22}$, the solution φ_i therefore becomes

$$\int\frac{\mathrm{d}\varphi_i}{\sqrt{a_i-b_iF(\varphi_i)}}=\mathrm{d}_{ij}q_j+q_{0i} \qquad q_{0i}=\text{constant}. \tag{20}$$

Given (20) and (10), we have a kind of general solution of equation (1) as follows;

$$\int\frac{\mathrm{d}\varphi_i}{\sqrt{a_i-b_iF(\varphi_i)}}=\mathrm{d}_{ij}[f_j(a_{k\alpha}x_\alpha)+b_{j\alpha}x_\alpha]. \tag{21}$$

This result is quite interesting and useful. By choosing different $F(\varphi)$ and $f_j(a_{k\alpha}x_\alpha)$, we can construct many multidimensional solitons with this result.

3. Bäcklund transformations

The Bäcklund transformation is an effective method for solving the sine-Gordon equation, the Korteweg–de Vries equation, and some other equations [19, 20]. However using this method to directly solve the general scalar field equation (1) is still difficult. In this section we will discuss this probem with the canonical transformation.

Assuming the solutions of (9) in the form

$$\mathscr{P}_1=\pi_1f_1(\varphi_1)+\pi_2[f_1(\varphi_1)-f_2(\varphi_1,\varphi_2)]$$
$$\mathscr{P}_2=\pi_1g_1(\varphi_1)-\pi_2[g_1(\varphi_1)-g_2(\varphi_1,\varphi_2] \tag{22}$$

then (8) gives the relations

$$\frac{\partial\varphi_1}{\partial q_1}-\frac{\partial\varphi_2}{\partial q_1}=\frac{\partial\mathscr{P}_1}{\partial\pi_1}-\frac{\partial\mathscr{P}_1}{\partial\pi_2}=f_2(\varphi_1,\varphi_2)$$
$$\frac{\partial\varphi_1}{\partial q_2}+\frac{\partial\varphi_2}{\partial q_2}=\frac{\partial\mathscr{P}_2}{\partial\pi_1}+\frac{\partial\mathscr{P}_2}{\partial\pi_2}=g_2(\varphi_1,\varphi_2) \tag{23}$$

between φ_1 and φ_2, that is a general form of Bäcklund transformation. Here $f_2(\varphi_1, \varphi_2)$ and $g_2(\varphi_1, \varphi_2)$ obey the nonlinear equation (9). Substituting (22) into (9) yields the systems of equations

$$f_1(\varphi_1)\frac{\mathrm{d}g_1(\varphi_1)}{\mathrm{d}\varphi_1}=\frac{\mathrm{d}F(\varphi_1)}{\mathrm{d}\varphi_1} \qquad g_1(\varphi_1)\frac{\mathrm{d}f_1(\varphi_1)}{\mathrm{d}\varphi_1}=\frac{\mathrm{d}F(\varphi_1)}{\mathrm{d}\varphi_1} \tag{24}$$

$$\begin{aligned} g_1(\varphi_1)\left(\frac{\partial}{\partial\varphi_2}-\frac{\partial}{\partial\varphi_1}\right)f_2-g_2\frac{\partial f_2}{\partial\varphi_2}&=\frac{\mathrm{d}F(\varphi_2)}{\mathrm{d}\varphi_2}-\frac{\mathrm{d}F(\varphi_1)}{\mathrm{d}\varphi_1} \\ f_1(\varphi_1)\left(\frac{\partial}{\partial\varphi_2}+\frac{\partial}{\partial\varphi_1}\right)g_2-f_2\frac{\partial g_2}{\partial\varphi_2}&=\frac{\mathrm{d}F(\varphi_2)}{\mathrm{d}\varphi_2}+\frac{\mathrm{d}F(\varphi_1)}{\mathrm{d}\varphi_1}. \end{aligned} \tag{25}$$

The solution of (24) may be easily obtained as

$$f_1(\varphi_1)g_1(\varphi_1)=2F(\varphi_1)+C \qquad C=\text{constant}. \tag{26}$$

For the latter purpose we select $f_1(\varphi_1)$ and $g_1(\varphi_1)$ in the forms

$$f_1(\varphi_1)=\lambda_1\sqrt{\pm2F(\varphi_1)+C} \qquad g_1(\varphi_1)=\lambda_1^{-1}\sqrt{\pm2F(\varphi_1)+C} \qquad \lambda_1=\text{constant}. \tag{27}$$

Thus (22) and (8) give a solution of (1) as

$$\begin{aligned} \int \mathrm{d}\varphi_1 &= \int \frac{\partial\varphi_1}{\partial q_i}\,\mathrm{d}q_i = \int \frac{\partial \mathscr{P}_i}{\partial \pi_1}\,\mathrm{d}q_i = \int f_1(\varphi_1)\,\mathrm{d}q_1 + g_1(\varphi_1)\,\mathrm{d}q_2 \\ &= \int \sqrt{\pm2F(\varphi_1)+C}(\lambda_1\,\mathrm{d}q_1+\lambda_1^{-1}\,\mathrm{d}q_2) \end{aligned}$$

namely

$$\int \mathrm{d}\varphi_1/\sqrt{\pm2F(\varphi_1)+C}=\lambda_1 q_1+\lambda_1^{-1}q_2+q_0 \qquad q_0=\text{constant} \tag{28}$$

which is contained by (20).

Equation (25) is a system of nonlinear equations of $f_2(\varphi_1, \varphi_2)$ and $g_2(\varphi_1, \varphi_2)$. If we can obtain the solutions of (25), then we give the explicit form of the Bäcklund transformation (23); but to solve (25) is not a simple problem. Here we only consider a particular case, namely g_2 and f_2 take the forms

$$f_2=f_2(\varphi_1+\varphi_2) \qquad g_2=g_2(\varphi_1-\varphi_2). \tag{29}$$

Given (29), the direct calculation leads to

$$\frac{\partial f_2}{\partial\varphi_2}-\frac{\partial f_2}{\partial\varphi_1}=0 \qquad \frac{\partial g_2}{\partial\varphi_2}+\frac{\partial f_2}{\partial\varphi_2}=0. \tag{30}$$

Applying (30) to (25) yields

$$\begin{aligned} &g_2\frac{\partial f_2}{\partial\varphi_2}=\frac{\mathrm{d}F(\varphi_1)}{\mathrm{d}\varphi_1}-\frac{\mathrm{d}F(\varphi_2)}{\mathrm{d}\varphi_2} \qquad f_2\frac{\partial g_2}{\partial\varphi_2}=-\left[\frac{\mathrm{d}F(\varphi_1)}{\mathrm{d}\varphi_1}+\frac{\mathrm{d}F(\varphi_2)}{\mathrm{d}\varphi_2}\right] \\ &\frac{\partial}{\partial\varphi_2}(g_2f_2)=-2\frac{\mathrm{d}F(\varphi_2)}{\mathrm{d}\varphi_2} \end{aligned} \tag{31}$$

and

$$g_2\frac{\partial f_2}{\partial\varphi_1}=\frac{\mathrm{d}F(\varphi_1)}{\mathrm{d}\varphi_1}-\frac{\mathrm{d}F(\varphi_2)}{\mathrm{d}\varphi_2} \qquad f_2\frac{\partial g_2}{\partial\varphi_1}=\frac{\mathrm{d}F(\varphi_1)}{\mathrm{d}\varphi_1}+\frac{\mathrm{d}F(\varphi_2)}{\mathrm{d}\varphi_2}$$
$$\frac{\partial}{\partial\varphi_2}(g_2f_2)=2\frac{\mathrm{d}F(\varphi_1)}{\mathrm{d}\varphi_1}. \tag{32}$$

Combining (31) with (32), we derive a specific solution of (25) which satisfies

$$f_2(\varphi_1+\varphi_2)\cdot g_2(\varphi_1-\varphi_2)=2F(\varphi_1)-2F(\varphi_2). \tag{33}$$

Equation (33) implies a few useful things. The following are several obvious examples.

(*a*) *The Klein–Gordon equation.* Setting $F(\varphi_i)=\frac{1}{2}\varphi_i^2$, (1) becomes a KG equation. From (33) and the equation

$$2F(\varphi_1)-2F(\varphi_2)=\varphi_1^2-\varphi_2^2=(\varphi_1+\varphi_2)(\varphi_1-\varphi_2)$$

we may take it that

$$f_2(\varphi_1+\varphi_2)=\lambda_a(\varphi_1+\varphi_2) \qquad g_2(\varphi_1-\varphi_2)=\lambda_a^{-1}(\varphi_1-\varphi_2) \qquad \lambda_a=\text{constant}. \tag{34}$$

(*b*) *The Liouville equation.* Since $F(\varphi_i)=\mathrm{e}^{\varphi_i}$ and

$$2F(\varphi_1)-2F(\varphi_2)=2(\mathrm{e}^{\varphi_1}-\mathrm{e}^{\varphi_2})=2\,\mathrm{e}^{(\varphi_1+\varphi_2)/2}[\mathrm{e}^{(\varphi_1-\varphi_2)/2}-\mathrm{e}^{-(\varphi_1-\varphi_2)/2}]$$
$$=4\,\mathrm{e}^{(\varphi_1+\varphi_2)/2}\sinh[\tfrac{1}{2}(\varphi_1-\varphi_2)]$$

so (33) gives

$$f_2=2\lambda_b\,\mathrm{e}^{(\varphi_1+\varphi_2)/2} \qquad g_2=2\lambda_b^{-1}\sinh[\tfrac{1}{2}(\varphi_1-\varphi_2)] \qquad \lambda_b=\text{constant}. \tag{35}$$

(*c*) *The sinh–Gordon equation.* In this case we have

$$2F(\varphi_1)-2F(\varphi_2)=2(\cosh\varphi_1-\cosh\varphi_2)=4\sinh[\tfrac{1}{2}(\varphi_1+\varphi_2)]\sinh[\tfrac{1}{2}(\varphi_1-\varphi_2)]$$

namely

$$g_2=2\lambda_c^{-1}\sinh[\tfrac{1}{2}(\varphi_1-\varphi_2)] \qquad f_2=2\lambda_c\sinh[\tfrac{1}{2}(\varphi_1+\varphi_2)] \qquad \lambda_c=\text{constant} \tag{36}$$

(*d*) *The sine–Gordon equation.* The case implies

$$2F(\varphi_1)-2F(\varphi_2)=2(\cos\varphi_2-\cos\varphi_1)=4\sin[\tfrac{1}{2}(\varphi_1+\varphi_2)]\sin[\tfrac{1}{2}(\varphi_1-\varphi_2)]$$
$$g_2=2\lambda_d^{-1}\sin[\tfrac{1}{2}(\varphi_1-\varphi_2)] \qquad f_2=2\lambda_d\sin[\tfrac{1}{2}(\varphi_1+\varphi_2)] \qquad \lambda_d=\text{constant}. \tag{37}$$

Inserting (34)-(37) respectively into (23), we obtain the corresponding Bäcklund transformations (BT). These BT will give some general soliton solutions of corresponding equations. In particular, by applying the theorem of commutability of BT, we can give some formulae of nonlinear superposition. For example, we have the well known formula [21]

$$\tan\frac{\varphi_3-\varphi_0}{4}=\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\tan\frac{\varphi_1-\varphi_2}{4} \qquad \lambda_1,\lambda_2=\text{constant} \tag{38}$$

where φ_i for $i=0, 1, 2, 3$ are the solutions of SG.

For the SG equation, (28) contains the general soliton solution

$$\varphi_1=4\tan^{-1}\exp(\lambda_1q_1+\lambda_1^{-1}q_2+q_0) \qquad q_0=\text{constant}. \tag{39}$$

Setting $\varphi_0=0$, $\varphi_2=4\tan^{-1}\exp(\lambda_2 q_1+\lambda_2^{-1}q_2+q_0')$, $q_0'=$ constant, and substituting them into (38), we obtain another general soliton solution of SG as

$$\varphi_3=4\tan^{-1}\left[\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\cdot\frac{e^{\lambda_1 q_1+\lambda_1^{-1}q_2+q_0}-e^{\lambda_2 q_1+\lambda_2^{-1}q_2+q_0'}}{1+e^{(\lambda_1+\lambda_2)q_1+(\lambda_1^{-1}+\lambda_2^{-2})q_2+q_0+q_0'}}\right] \tag{40}$$

where q_1 and q_2 are given by (10). Further we can obtain N general soliton solutions of the multidimensional SG, by repeatedly using the superposition formula (38). For the Klein–Gordon equation, the Liouville equation and the sinh–Gordon equation, we may manifestly find similar results, since we have given their Bäcklund transformation.

In general, the Bäcklund transformation depends on the solutions of (24) and (25). For various nonlinear scalar field equations with different $F(\varphi)$ the solution to (25) will be an interesting exercise.

4. Multidimensional solitons and breathers

Let us now take (39) and (40) as examples to discuss some properties of the multidimensional solitons for the SG equation. First we simply show that the general soliton solutions (39) and (40) include some well known results. Let us look back at (10) and (12). According to (12) we may take all of $a_{k\alpha}$ to be equal to zero and $b_{10}=i/2$, $b_{11}=1/2$, $b_{20}=-i/2$, $b_{21}=1/2$ and $b_{1\iota}=b_{2\iota}=0$ for $i>1$. Substituting them into (10) gives

$$q_1=\tfrac{1}{2}(x+t)+\delta_1 \qquad q_2=\tfrac{1}{2}(x-t)+\delta_2 \qquad \delta_1,\delta_2=\text{constant}. \tag{41}$$

In this case, (39) and (40) become well known (1+1) dimensional solitons of SG. On the other hand, setting

$$q_\iota=\ln\sum_{k=1}^{N}e^{\eta_k}+b_{\iota\alpha}x_\alpha=\ln\left(e^{b_{\iota\alpha}x_\alpha}\sum_{k=1}^{N}e^{a_{k\beta}x_\beta+\delta_k}\right)$$
$$=\ln\sum_{k=1}^{N}\exp[(a_{k\alpha}+b_{i\alpha})x_\alpha+\delta_k] \qquad \delta_k=\text{constant} \tag{42}$$

in (10), then (39) and (40) denote some multiple solitons of multidimensional SG, that is the Gibbon's results [9, 10]. We will discuss some new interesting multidimensional solitons in detail.

4.1. *The (2+1)-dimensional solitons*

When $n=4$, (13) restricts the number N of variables η_k to $N\leqslant 5$. We only consider the simple case $N=1$. In the case, (12) contains six equations as

$$a_{10}^2+a_{11}^2+a_{12}^2+a_{13}^2=0 \qquad b_{10}^2+b_{11}^2+b_{12}^2+b_{13}^2=0 \qquad b_{20}^2+b_{21}^2+b_{22}^2+b_{23}^2=0$$
$$a_{10}b_{10}+a_{11}b_{11}+a_{12}b_{12}+a_{13}b_{13}=0 \qquad a_{10}b_{20}+a_{11}b_{21}+a_{12}b_{22}+a_{13}b_{23}=0 \tag{43}$$
$$b_{10}b_{20}+b_{11}b_{21}+b_{12}b_{22}+b_{12}b_{23}=\tfrac{1}{2}$$

with twelve constants. Therefore (43) implies the existance of six arbitrary constants. We select a group of simple solutions as follows;

$$a_{10}=i \qquad a_{11}=1 \qquad a_{12}=a_{13}=0$$
$$b_{12}=1/2 \qquad b_{13}=i/2 \qquad b_{10}=b_{11}=0 \tag{44}$$
$$b_{22}=1/2 \qquad b_{23}=-i/2 \qquad b_{20}=b_{21}=0.$$

Applying (44) to (10) gives the field canonical variables

$$q_1 = f_1(x-t) + y/2 + iz/2 \qquad q_2 = f_2(x-t) + y/2 - iz/2. \tag{45}$$

Given (45), from (39) we have a general soliton

$$\varphi = 4\tan^{-1}\exp[f(x-t) + \tfrac{1}{2}\lambda_1(y+iz) + \tfrac{1}{2}\lambda_1^{-1}(y-iz)] \tag{46}$$

where $f(x-t) = \lambda_1 f_1(x-t) + \lambda_1^{-1} f_2(x-t) + q_0$ is an arbitrary function of $(x-t)$. Taking $\lambda_1 = \lambda_1^{-1} = 1$, (46) becomes a (2+1)-dimensional general soliton solution

$$\varphi = 4\tan^{-1}\exp[f(x-t) + y] = 4\tan^{-1}\exp\xi \tag{47}$$

where

$$\xi = f(x-t) + y. \tag{48}$$

Although (47) is a simple form it can describe many interesting solitons. For any definite ξ, one hand φ has the corresponding definite value, on the other hand (48) denotes a general plane curve which moves along the x direction. The image of curve (48) determines the shape of soliton (47). There are many instances of (2+1) dimensional solitions such as:

A. The parabola soliton

$$\varphi_a = 4\tan^{-1}\exp[(x-t)^2 + y].$$

B. The catenary soliton

$$\varphi_b = 4\tan^{-1}\exp[\mathrm{ch}(x-t) + y].$$

C. The tractrix soliton

$$\varphi_c = 4\tan^{-1}\exp[\mathrm{Arcch}(x-t)^{-1} + \sqrt{1-(x-t)^2} + y].$$

D. The Gaussian curve soliton

$$\varphi_d = 4\tan^{-1}\exp[e^{-(x-t)^2/2} + y].$$

E. The hypocycloid soliton

$$\varphi_e = 4\tan^{-1}\exp\{(1-(x-t)^{2/3}]^{3/2} + y\}.$$

F. The cycloid soliton

$$\varphi_f = 4\tan^{-1}\exp[\sqrt{(x-t)(2-x+t)} - \cos^{-1}(1-x+t) + y].$$

G. The strophoid soliton

$$\varphi_g = 4\tan^{-1}\exp[(x-t)^2\sqrt{(2-x+t)/(1+x-t)} + y].$$

H. The conical section soliton $\varphi_h = 4\tan^{-1}\exp[\sqrt{A^2 - B^2(x-t)^2} + y]$, A, B = constant and $B^2 > 0$ which makes φ_h the ellipsoid soliton and $B^2 < 0$ the hyperboloid.

Setting $\varphi = z$, then any of these solitons is formed by (piling) an infinite number of corresponding curves on planes $z = z_i \leqslant (\pi/2)$ $(i = 1, 2, \ldots, \infty)$. For example, the φ_h seem to be an upside-down pipe which is piled by an infinite number of curves $\xi_i = \sqrt{A^2 - B^2(x-t)^2} + y$ or $(y-\xi_i)^2 + B^2(x-t)^2 = A^2$ on $z = z_i \leqslant (\pi/2)$ $(i = 1, 2, \ldots, \infty)$ planes. Here ξ_i, A and B are real constants; they determine the forms and places of the conical section, therefore the shapes of the soliton φ_h.

4.2. The (3+1)-dimensional solitons

We have known that the (2+1)-dimensional soliton (47) has some values the same on the corresponding general plane curves. Similarly, the (3+1)-dimensional solitons have the same values on some general curved surfaces. Let us see a simple example. Taking a group of solitions of (12) as

$$\begin{aligned} &a_{10}=i \qquad a_{11}=1 \qquad a_{1i}=a_{jk}=0 \qquad i,j>1 \\ &b_{12}=b_{13}=1/(2\sqrt{2}) \qquad b_{14}=i/2 \qquad b_{10}=b_{11}=b_{1i}=0 \qquad i>4 \\ &b_{22}=b_{23}=1/(2\sqrt{2}) \qquad b_{24}=-i/2 \qquad b_{20}=b_{21}=b_{2i}=0 \qquad i>4 \end{aligned} \tag{48}$$

and inserting these into (10) gives

$$\begin{aligned} q_1&=f_1(x-t)+(y+z)/(2\sqrt{2})+ix_4/2 \\ q_2&=f_2(x-t)+(y+z)/(2\sqrt{2})-ix_4/2. \end{aligned} \tag{49}$$

Application of (39) and (40) leads a solution

$$\varphi=4\tan^{-1}\exp[f(x-t)+(y+z)/\sqrt{2}]=4\tan^{-1}\exp\zeta \tag{50}$$

when $\lambda_1=1$, $f_1(x-t)+f_2(x-t)=f(x-t)$ and

$$\zeta=f(x-t)+(y+z)/\sqrt{2}. \tag{51}$$

For any definite ζ and time t, (51) denotes a general cylindrical surface. On the surface, the soliton solution (50) takes the same value. The shape of the soliton depends on the image of the surface. Setting

$$\begin{aligned} &p=\frac{\partial z}{\partial x}=-\sqrt{2}\frac{\partial f}{\partial x} \qquad q=\frac{\partial z}{\partial y}=-1 \qquad h=(1+p^2+q^2)^{1/2}=\left[2+2\left(\frac{\partial f}{\partial x}\right)^2\right]^{1/2} \\ &y=\frac{\partial^2 z}{\partial x^2}=-\sqrt{2}\frac{\partial^2 f}{\partial x^2} \qquad s=\frac{\partial^2 z}{\partial x\partial y}=0 \qquad t=\frac{\partial^2 z}{\partial y^2}=0 \end{aligned} \tag{52}$$

we have the first fundamental quantities [22]

$$E=1+p^2=1+2\left(\frac{\partial f}{\partial x}\right)^2 \qquad F=pq=\sqrt{2}\frac{\partial f}{\partial x} \qquad G=1+q^2=2 \tag{53}$$

the second ones

$$L=\frac{r}{h}=\frac{\partial^2 f/\partial x^2}{\sqrt{1+(\partial f/\partial x)^2}} \qquad M=\frac{s}{h}=N=\frac{t}{h}=0 \tag{54}$$

and the corresponding fundamental forms

$$\omega_1^2=\left[1+2\left(\frac{\partial f}{\partial x}\right)^2\right]\mathrm{d}x^2+2\sqrt{2}\frac{\partial f}{\partial x}\mathrm{d}x\,\mathrm{d}y+2\,\mathrm{d}y^2 \tag{55}$$

$$\omega_2^2=-\frac{\partial^2 f/\partial x^2}{\sqrt{1+(\partial f/\partial x)^2}}\mathrm{d}x^2. \tag{56}$$

Given (52), we can also obtain the principal radius of curvature

$$R=-\frac{h^3}{(1+q^2)\gamma}=\frac{[1+(\partial f/\partial x)^2]^{3/2}}{\partial^2 f/\partial x^2}. \tag{57}$$

Explicitly, if the function f takes the form

$$f=\sqrt{a^2-(x-t)^2} \qquad a=\text{arbitrary constant} \tag{58}$$

the curvature R is then equal to the constant a, and the cylindrical surface (51) becomes a circular one. By substituting (58) into (50), we obtain a cylindrical surface soliton which moves along the x direction.

In order to construct other (3+1)-dimensional solitons, let us rewrite (45) as

$$\begin{aligned} q_1&=[f_1(x-t)+y]/2+i[f_2(x-t)+z]/2 \\ q_2&=[f_1(x-t)+y]/2-i[f_2(x-t)+z]/2. \end{aligned} \tag{59}$$

Therefore we have

$$\lambda_i q_1+\lambda_i^{-1}q_2=\tfrac{1}{2}(\lambda_i+\lambda_i^{-1})[f_1(x-t)+y]+\frac{i}{2}(\lambda_i-\lambda_i^{-1})[f_2(x-t)+z]. \tag{60}$$

Setting $\lambda_1+\lambda_1^{-1}=\lambda_2+\lambda_2^{-1}=a_1$, $\lambda_1-\lambda_1^{-1}=-(\lambda_2-\lambda_2^{-1})=a_2$ and inserting (60) into (40) yields

$$\varphi=4\tan^{-1}\frac{a_1\,\mathrm{e}^{a_1[f_1(x-t)+y]/2}\{\mathrm{e}^{ia_2[f_2(x-t)+z]/2+q_0}-\mathrm{e}^{-ia_2[f_2(x-t)+z]/2+q_0'}\}}{a_2\{\mathrm{e}^{a_1[f_1(x-t)+y]+q_0+q_0'}\}}. \tag{61}$$

Taking $q_0=-\pi i/2$, $q_0'=\pi i/2$ and $a_2=-ia_3$, (61) becomes another kind of general soliton solution

$$\varphi=4\tan^{-1}\frac{a_1\,\mathrm{ch}\{a_3[f_2(x-t)+z]\}}{a_3\,\mathrm{ch}\{a_1[f_1(x-t)+y]\}}. \tag{62}$$

By selecting different functions $f_1(x-t)$ and $f_2(x-t)$, (62) will give many interesting (3+1)-dimensional soliton solutions.

4.3. The (3+1)-dimensional breathers

Our discussion starts from equation (61). It is explicit that (61) contains the general breather solution

$$\varphi=4\tan^{-1}\frac{a_1\sin\{a_2[f_2(x-t)+z]/2\}}{a_2\sinh\{a_1[f_1(x-t)+y]/2\}} \tag{63}$$

when $q_0=q_0'=-\pi i/2$. This solution has vibratility and some other properties. Let a_2 be an infinitesimal quantity $a_2\to 0$. Then the definitions of a_1 and a_2 imply $a_1\approx 2$. In this case, (63) leads to

$$\varphi=\lim_{a_2\to 0}4\tan^{-1}\frac{a_1\sin\{a_2[f_2(x-t)+z]/2\}}{a_2\sinh\{a_1[f_1(x-t)+y]/2\}}=4\tan^{-1}\frac{f_2(x-t)+z}{\sinh[f_1(x-t)+y]}. \tag{64}$$

This is a new (3+1)-dimensional general soliton solution. Obviously setting $f_2(x-t)=\sin(x-t)$ yields a fresh general breather solution.

5. Stability of the multidimensional solitons

We now come to simply discussing the stability [23] of the multidimensional solitons by making use of the energy of the fields. The energy of the n-dimensional scalar field may be expressed as

$$H=\int \mathrm{d}^{n-1}x\{[\partial_i\varphi\partial_i\varphi-\partial_0\varphi\partial_0\varphi]/2+F(\varphi)\}. \tag{65a}$$

For the stable field, $\partial_0\varphi=0$ makes

$$H_0=\int \mathrm{d}^{n-1}x[\tfrac{1}{2}\partial_i\varphi\partial_i\varphi+F(\varphi)]=\int \mathrm{d}^{n-1}x\,\mathcal{H}[\varphi,\partial_i\varphi]. \tag{65b}$$

Combining (1) with (65*b*), we have the variation and second variation

$$\begin{aligned}\delta H_0&=\int \mathrm{d}^{n-1}x\left[\frac{\partial\mathcal{H}}{\partial\varphi}-\partial_i\frac{\partial\mathcal{H}}{\partial(\partial_i\varphi)}\right]\delta\varphi=\int \mathrm{d}^{n-1}x\left(\frac{\mathrm{d}F}{\mathrm{d}\varphi}-\partial_i\partial_i\varphi\right)\delta\varphi=0\\ \delta^2H_0&=\int \mathrm{d}^{n-1}x\left[\frac{\mathrm{d}^2F}{\mathrm{d}\varphi^2}\delta\varphi^2+\frac{\partial^2\mathcal{H}}{\partial(\partial_i\varphi)^2}(\delta\partial_i\varphi)^2\right]=\int \mathrm{d}^{n-1}x\left[\frac{\mathrm{d}^2F}{\mathrm{d}\varphi^2}\delta\varphi^2+(\partial_i\delta\varphi)(\partial_i\delta\varphi)\right].\end{aligned} \tag{66}$$

The principle of least energy [24] shows that

$$\delta^2H_0>0 \quad \text{or} \quad \frac{\mathrm{d}^2F}{\mathrm{d}\varphi^2}>0 \tag{67}$$

leads the energy to minimum and, therefore the soliton to stabilization.

First we consider the SG equation with $F(\varphi)=-\cos\varphi$. Since $\mathrm{d}^2F/\mathrm{d}\varphi^2=\cos\varphi$, (67) requires that

$$-\pi/2\leqslant\varphi\leqslant\pi/2. \tag{68}$$

Inserting (39) into (68) yields

$$-\tan\pi/8\leqslant\exp(\lambda_1q_1+\lambda_1^{-1}q_2+q_0)\leqslant\tan\pi/8. \tag{69}$$

In this region, the soliton solution (39) is stable.

We then study the φ^4 field equation. In the case, applying $F(\varphi)=\frac{1}{2}a\varphi^2-\frac{1}{4}b\varphi^4$ to (20) gives a general soliton solution

$$A\gamma\,\mathrm{sech}(\sqrt{b/2a}\,\varphi)=ad_jq_j+q_0 \qquad j=1,2. \tag{70}$$

This solution has the property

$$0\leqslant\sqrt{b/2a}\,\varphi=\mathrm{sech}(ad_jq_j+q_0)\leqslant1. \tag{71}$$

Therefore we have $\varphi_{\max}=\sqrt{2a/b}$ and

$$\frac{\mathrm{d}^2F}{\mathrm{d}\varphi^2}=a-3b\varphi^2\geqslant a-3b\varphi_{\max}^2=(1-6)a=-5a. \tag{72}$$

Equation (72) makes the condition of stabilization the following

$$a<0. \tag{73}$$

Under the condition (73), the multidimensional solitons of φ^4 field is stable. This is an important result.

6. Conclusion

We presented the general method of canonical transformation for solving scalar field equations. By making use of this method, we derived a general form of Bäcklund transformation and some general soliton and breather solutions of the equations. In particular, the Bäcklund transformation furnishes a way to construct N general soliton solutions of the SG equation. We showed that these general solutions include many interesting soliton specific solutions. Further we discussed the properties of the multi-dimensional solitons and obtained the conditions of stabilization for these solitons.

We believe that the canonical transformation is also effective for solving other nonlinear field equations and the general soliton solutions will have quite widespread applications in the actual physical problems.

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